B.Sc. Part-II (H) Paper-III 2.7 Calculus 30-07-2020
Parametric curve.

Q.1. Find the radius of curvature of the curve
$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$
(I) at the point θ (II) at the point $\theta = 0$

Soln: The given curves are
$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$
$\therefore$ Diff. w.r.t. to θ, we get
$$\frac{dx}{d\theta} = a(1 + \cos\theta),\quad \frac{dy}{d\theta} = a\sin\theta$$
Again, diff. w.r.t. to θ, we get
$$\frac{d^2x}{d\theta^2} = -a\sin\theta,\quad \frac{d^2y}{d\theta^2} = a\cos\theta$$
(I) Now, we have the ~~formula~~ formula.
$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}$$
$$= \frac{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]^{3/2}}{a^2\cos\theta(1+\cos\theta) + a^2\sin^2\theta} = \frac{a^3(2+2\cos\theta)^{3/2}}{a^2(1+\cos\theta)}$$
$$= 2^{3/2}a(1+\cos\theta)^{\frac{1}{2}} = 2^{3/2}\cdot a\cdot 2^{\frac{1}{2}}\cos\frac{\theta}{2} = 4a\cos\frac{\theta}{2}$$

and so (II) at $\theta = 0$, $\rho = 4a$
solved

Q.2. Show that the radii of curvature at the origin for the curve
$x^3 + y^3 = 3axy$ are each $= \frac{3a}{2}$

Solution:→ We know that the tangents to the curve at the origin are obtained by equating to zero the lowest degree terms in the equation of the curve.

Since there is no constant term in the equation of the curve, therefore the curve passes through the origin.

Hence the tangents to the curve at the origin are given by
$$3axy = 0$$
That is, either $x = 0$, that is, the y-axis
or $y = 0$ that is, the x-axis.

Thus both the axes are tangents to the curve at the origin. 30-07-2020

Case I. When the x-axis is tangent to the curve at the origin, then by the formula for the Newtonian method,

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y}$$

Now, $x^3+y^3 = 3axy$

Dividing this by $2xy$, we get

$$\frac{x^3}{2xy} + \frac{y^3}{2xy} = \frac{3axy}{2xy}$$

$$\Rightarrow \frac{x^2}{2y} + \frac{1}{4} x \cdot y \cdot \frac{2y}{x^2} = \frac{3a}{2}$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \left(\frac{x^2}{2y} + \frac{1}{4} xy \frac{2y}{x^2}\right) = \frac{3a}{2}$$

$$\Rightarrow \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2}{2y} + \frac{1}{4} \lim_{x\to 0} x \lim_{y\to 0} y \lim_{\substack{x\to 0\\ y\to 0}} \frac{2y}{x^2} = \frac{3a}{2}$$

$$\Rightarrow \rho + \frac{1}{4} \cdot 0 \cdot 0 \cdot \frac{1}{\rho} = \frac{3a}{2}$$

$$\Rightarrow \rho = \frac{3a}{2}$$

Case II. When the y-axis is tangent to the curve at the origin, then

$$\rho = \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}$$

Now, $x^3+y^3 = 3axy$

Dividing both the sides by $2xy$, we get

$$\frac{x^3}{2xy} + \frac{y^3}{2xy} = \frac{3axy}{2xy}$$

$$\Rightarrow \frac{x^2}{2y} + \frac{y^2}{2x} = \frac{3a}{2}$$

$$\Rightarrow \frac{1}{4} \frac{xy}{\frac{y^2}{2x}} + \frac{y^2}{2x} = \frac{3a}{2}$$

$$\Rightarrow \frac{1}{4} \cdot \frac{\left(\lim_{x\to 0} x\right)\left(\lim_{y\to 0} y\right)}{\lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x}} + \lim_{\substack{x\to 0\\ y\to 0}} \frac{y^2}{2x} = \frac{3a}{2}$$

$$\Rightarrow \frac{1}{4} \cdot \frac{0 \cdot 0}{\rho} + \rho = \frac{3a}{2}$$

Hence, $\rho = \frac{3a}{2}$ Proved